Bilinear Hilbert Transform - 01/06/2015.

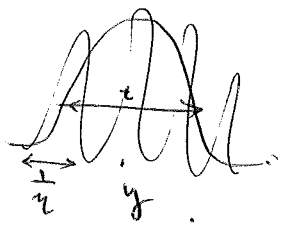
$$(f_1,f_2) \longmapsto \text{p.v.} \int_{\mathbb{R}} f_1(x-t)\, f_2(x+t)\, \frac{dt}{t}.$$

$$L^{p_1}\times L^{p_2} \longrightarrow L^{p}$$

$$\Lambda(f_1,f_2,f_3) := \text{p.v.} \iint \prod_{j=1}^{3} f_j(x-\beta_j t)\, dx\, \frac{dt}{t}.$$

"Wave packet transform":

$$F(y,\eta,t) = \int_{\mathbb{R}} f(x)\, e^{i\eta(y-x)}\, \frac{1}{t}\varphi\left(\frac{y-x}{t}\right) dx.$$



For each f_i, consider transformed function; and

$$\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_j \underbrace{F_j\left(y, \alpha_j\eta+\beta_j\tfrac{1}{t}, t\right)} . \, dy\, d\eta\, dt \longleftarrow \underline{\underline{\text{not}}}\ \frac{dt}{t}.$$

$$\downarrow$$

$$\int_{\mathbb{R}} f_j(x_j)\, e^{i(\alpha_j\eta+\beta_j\frac{1}{t})(y-x_j)}\, \frac{1}{t}\varphi\left(\frac{y-x_j}{t}\right) dx_j$$

<u>Recall</u>: $(1,1,1)$, $(\beta_1,\beta_2,\beta_3)$, $(\alpha_1,\alpha_2,\alpha_3)$ ort. in $\mathbb{R}^3$.

and since $e^{i(\alpha_j\eta+\beta_j\frac{1}{t})y}$ is ~~not~~ out in $\int dx_j$, we can get rid of this term

$$\iint dy\, dt \int d\eta\, \underbrace{\int d\bar{x}\, e^{i(\alpha\eta,\bar{x})} \cdot \underbrace{\prod_j f_j(x_j)\, e^{i\beta_j\frac{1}{t}x_j}\, \frac{1}{t}\varphi\left(\frac{y-x_j}{t}\right)}_{g(\bar{x})}}_{\hat{g}(\alpha\eta).}$$

by rot. invariance of F.T.

$$\int \hat{g}(\eta\alpha)\,d\eta = \iint g(\eta(1,1,1)+v\beta)\,d\eta\,dv$$

$$= \iint dy\,dt \iint d\eta\,dv.$$

$$\prod_j f_i(\eta+\beta_i v)\,\underbrace{e^{-i\beta_i\frac{1}{t}\cdot(\eta+\beta_i v)}}_{e^{-iv/t}\ \text{assuming } \|\beta\|=1}\,\frac{1}{t}\varphi\left(\frac{y-\eta-\beta_i v}{t}\right)$$

only place where y appears.

$$\int dy \prod_j \varphi\left(\frac{y-\eta-\beta_i v}{t}\right) \quad \left(\frac{y-\eta}{t}=z\right).$$

$$= \int t\,dz \prod_j \varphi\left(z-\beta_i\frac{v}{t}\right) =: t\,\psi\left(\frac{v}{t}\right).$$

Back to main int:

$$= \int dt \int dv \underbrace{\left(\int d\eta \left(\prod_j f_i(\eta+\beta_i v)\right) e^{-iv/t}\right)}_{h(v).} \cdot \left(\frac{1}{t^2}\cancel{t}\right)\psi\left(\frac{v}{t}\right).$$

$$\left(\begin{matrix} v'=v \\ t=v/t \end{matrix} \quad \left|\frac{d(v',t')}{d(v,t)}\right| = \begin{vmatrix} 1 & 0 \\ 0 & v/t^2 \end{vmatrix} = \frac{v}{t^2} = (t')^2\frac{1}{v'} \right).$$

$$= \iint dt'\,dv' \cdot \frac{\cancel{v'}}{\cancel{(t')^2}} h(v')\,e^{-it'}\,\cancel{(t')^2}\cdot\frac{1}{(v')^{\cancel{2}}}\psi(t')$$

$$= \int h(v)\frac{1}{v}\,dv \cdot \cancel{\hat{\psi}(1)}\; e^{it}\psi(t).$$

$$= \int h(v)\frac{1}{v}\,dv \cdot \underbrace{\hat{\psi}(1)}_{>0}.$$

>0 by assuming that bump funct. is >0.

$$=: c \cdot \int \frac{dv}{v} \int dx \, \Pi_j f_j(x + \beta_j v) \; = c\Lambda(f_1, f_2, f_3).$$

Rigorously there should be error term (in paper), which can be handled easily via Hölder.

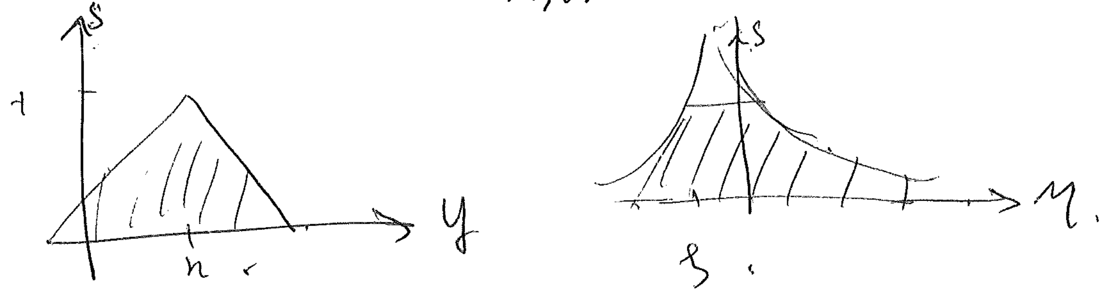
Need:

$$\left| \iiint \Pi_j F_j(y, \alpha_j \eta + \beta_j \tfrac{1}{t}, t) \, dy \, d\eta \, dt \right| \lesssim \Pi_j \|f_j\|_{L^{p_j}(\mathbb{R})}.$$

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad p_j > 2.$$

1. Outer estimate of linear functional ~~$S_1(f, E$~~

$$S_1(f, E) = \frac{1}{t} \iiint_{T(x, \xi, t)} |f| \, dy \, d\eta \, ds$$



Premeasure $\sigma(T) = t$.

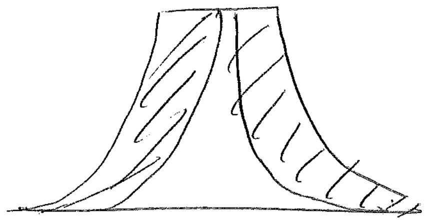
$$\Rightarrow \left| \iiint f \, dy \, d\eta \, ds \right| \lesssim \|f\|_{L^1(\cancel{\mathbb{R}^3_+}, \sigma, S_1)}$$

2. Outer Hölder est.

$$\lesssim \Pi_j \|F_j\|_{L^{p_j}(\mathbb{R}^3_+, \sigma, S_j)}.$$

find what size is appropriate.

Need: $\cancel{S_3}\, S_1(f_1 f_2 f_3, T) \lesssim \prod_j S_j(f_j, T)$.



Need S_j sizes using non-solid tents:

"Annulus tents."

$$T^{(j)}(x,\xi,t) = \{(y,\eta,s) : 0<s<t,\ |y-x|<t-s,$$
$$\left|(\eta-\xi)-\beta_j \tfrac{1}{s}\right| \le b\,|\alpha_j|\,\tfrac{1}{t}\}.$$

b chosen small enough >0.

$\{T^{(j)}\}_j$ are pairwise disjoint.

$$|\beta_1-\beta_2| \le s\left|(\eta-\xi-\beta_1\tfrac{1}{s})-(\eta-\xi-\beta_2\tfrac{1}{s})\right|$$
$$\le \cancel{s}\, b(|\alpha_1|+|\alpha_2|)\,\cancel{\tfrac{1}{s}}$$

Assume β_j are distinct.

$$S(f_1f_2f_3,T) \le \frac{1}{t}\int_{T\setminus(T^1\cup T^2\cup T^3)}|f_1f_2f_3| + \sum_j \frac{1}{t}\int_{T\cap T_j}|f_1f_2f_3|$$
$$\le \frac{1}{t}\prod_j\left(\int_{T\setminus T^j}|f_j|^3\right)^{\frac13} + \sum_j \sup_T \prod_{k\neq j}\left(\int_T|f_k|^2\right)^{\frac12}.$$
$$\lesssim \prod_j \underbrace{\left(\left(\frac{1}{t}\int_{T\setminus T^j}|f_j|^2\right)^{\frac12} + \sup_T|f_j|\right)}_{S_j(f_j,T)}.$$

by disjointness of $\{T^j\}$.

$\underline{\text{Th}^{\underline{n}}}$. $\| F(x,\xi,t)\|_{L^p(x,\sigma,S^b)} \lesssim \|f\|_{L^p(\mathbb{R})}$

for. $2 < p \leq \infty$ and $S^b(F,T) := \frac{1}{t}\left(\int_{T_{\alpha\beta}\setminus T^b} |F|^2 \, dy\, d\eta\, ds\right)^{\frac{1}{2}} + \sup_{T_{\alpha\beta}} |F|$.

$T_{\alpha\beta} =. \left|\alpha(\eta-\xi) + \frac{1}{\beta}\frac{1}{s}\right| \leq \frac{1}{s}$.

$T^b : |\eta - \xi| \leq b\frac{1}{s}$

$\underline{\text{Today}}$: $p = \infty$. Need: $\underbrace{\operatorname{outsup}_X S^b(F)}_{\sup_T S^b(F,T(x,\xi,t))} \lesssim \|F\|_\infty$.

$\|F\|_\infty \lesssim \|f\|_\infty \cdot \|\varphi_{x,\xi,t}\|_1 \leq C < \infty$.

Need: $\int_{T_{\alpha\beta}\setminus T^b} |F|^2 \, dy\, d\eta\, ds \lesssim \|f\|_2$

$\text{LHS} \lesssim \int_0^\infty ds \int_{\mathbb{R}} dy \int_{b/s \leq |\eta-\xi| \leq C/s} d\eta\, |F|^2$. $\quad (\eta - \xi = r\frac{1}{s})$.

$\lesssim \int_0^\infty ds \int_{\mathbb{R}} dy . \int_0^C dr\, \underbrace{|F(y, \xi + r\tfrac{1}{s}, s)|^2}_{\int f(x) \underbrace{e^{i(\xi + r\frac{1}{s})(y-x)} \frac{1}{s}\varphi\left(\frac{y-x}{s}\right)}_{\varphi_{y,r,s}(x)} dx}$.

$\lesssim \sup_{r \in (b,C)} A$.

$A^2 =. \left(\int_0^\infty \frac{ds}{s} \int_{\mathbb{R}} dy\, \underbrace{|\langle f, \varphi_{y,r,s}\rangle|^2}_{\langle \langle f, \varphi_{y,r,s}\rangle \varphi_{y,r,s}, f\rangle}\right)^2$

$$\lesssim \left\| \int_0^\infty \frac{ds}{s} \int_{\mathbb{R}} dy \, \langle f, \varphi_{y,r,s} \rangle \varphi_{y,r,s} \right\|_2^2 .$$

$$= \int_0^\infty \frac{ds}{s} \int_{\mathbb{R}} dy \int_0^\infty \frac{dr}{r} \int_{\mathbb{R}} dz \, | \langle f, \varphi_{y,r,s} \rangle \langle \varphi_{y,r,s} ; \varphi_{z,r,r} \rangle \langle f, \varphi_{z,r,r} \rangle |$$

(w.l.o.g. $|\langle f, \varphi_{z,r,r} \rangle| \leq |\langle f, \varphi_{y,r,s} \rangle|$).

$$\lesssim \int_0^\infty \frac{ds}{s} \int_{\mathbb{R}} dy \, |\langle f, \varphi_{y,r,s} \rangle|^2 .$$

$$\left(\int_0^\infty \frac{dr}{r} \int_{\mathbb{R}} dz \, |\langle \varphi_{y,r,s}, \varphi_{z,r,r} \rangle| \right)$$

(Schur est).

spt $\hat{\varphi} \subset [-\varepsilon, \varepsilon]$, spt $\widehat{\frac{1}{s}\varphi\left(\frac{y-\cdot}{s}\right)} \subset \left[-\frac{\varepsilon}{s}, \frac{\varepsilon}{s}\right]$,

spt $\underbrace{\widehat{e^{i\xi \cdot} \varphi_{y,r,s}}}_{\int = 0}$ $\subset \left[-\frac{\varepsilon}{s} + \frac{r}{s}, \frac{\varepsilon}{s} + \frac{r}{s}\right] \not\ni 0$.

So in integral: under $\int_{r \geq s} + \int_{r \leq s}$.

Case $r \geq s$:

$\varphi_{y,r,s} e^{i\xi(\cdot)}$, Schwarz $\int = 0$, scale s,

L^∞ norm $\frac{1}{s}$, pos. y

primitive: scale s, L^∞-norm 1 – pos y.

derivative: —"— – L^∞ norm $\frac{1}{s^2}$ —"—.

$\to$ IBP $\Rightarrow$.

$$\int_{\mathbb{R}} dz \int_s^\infty \frac{dr}{r} \int \frac{1}{\left(1+\frac{|y-x|}{s}\right)^2} \cdot \frac{\frac{1}{r^2}}{\left(1+\frac{|z-x|}{r}\right)^2} \cdot dx .$$

$$\lesssim \int_s^\infty \frac{dr}{r^2} \underbrace{\int_{\mathbb{R}} \frac{1}{\left(1+\frac{|y-x|}{s}\right)^2}}_{\lesssim s .} \cdot \lesssim 1 .$$